Prof. Banerjee's Practice MT Solns

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Spring 2016 Math 2E

Problem 1

a) Find the Jacobian of the transformation $x = \ln(uv),\ y = u^2 v$.

$\rightarrow$

$$J = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det \begin{bmatrix} \frac{v}{uv} & \frac{u}{uv} \\ 2uv & u^2 \end{bmatrix} = u - 2u = -u.$$

But, note that if we were to use the Jacobian in an integral, we would use $|J| = |-u|$.

b) Show that if $\nabla f = \nabla g$, then $f = g + c$ where c is a constant.

$\rightarrow$ We see $\nabla(f - g) = 0$ then. But only gradients of constants yield zero so hence $f - g = c$ where c is a constant, and rearranging, $f = g + c$.

Problem 2

a) Convert from spherical to rectangular coordinates the point $(2, \pi/4, \pi/3)$.

$\rightarrow$ It's not specified, but I think $\rho = 2, \theta = \pi/4, \phi = \pi/3$. Then,

$$\begin{cases} x = 2\cos\pi/4 \sin\pi/3 = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \sqrt{\frac{3}{2}}. \\ y = 2\sin\pi/4 \sin\pi/3 = (same) = \sqrt{\frac{3}{2}}. \\ z = 2\cos\pi/3 = 2 \cdot \frac{1}{2} = 1. \end{cases}$$

b) Express in cylindrical coordinates $y + z = 2z^2$.
$\rightarrow$ In cylindrical, z doesn't change, but $y = r\sin\theta$ so we have $r\sin\theta + z = 2z^2$.

If necessary, we can complete the square: $y = 2z^2 - z = 2(z - \frac{1}{4})^2 - \frac{1}{8}$ so then $2(z - \frac{1}{4})^2 = r\sin\theta + \frac{1}{8}$. Then we see $z = \frac{1}{4} \pm \sqrt{\frac{r}{2}\sin\theta + \frac{1}{16}}$. But I think this shouldn't be necessary...

Problem 3

Find the volume between the paraboloid $z = 6 - x^2 - y^2$ and the cone $z = \sqrt{x^2 + y^2}$.

$\rightarrow$ The two surfaces intersect as a circle (if we draw it). To find the curve of intersection, one way is to substitute $z = \sqrt{x^2 + y^2} \iff x^2 + y^2 = z^2$ into the paraboloid, and we see

$$z = 6 - z^2 \iff z^2 + z - 6 = 0 \iff (z+3)(z-2) = 0,\ z = 2, -3.$$

They must intersect at $z = 2$, not the negative one. And, at $z = 2$, we see we have $2 = \sqrt{x^2 + y^2} \iff x^2 + y^2 = 4$, a circle of radius two. Hence, our xy domain is everything inside this circle. We can now set up the volume,

$$V = \iint_{Disc\ of\ Radius\ 2} \int_{z=\sqrt{x^2+y^2}}^{6-x^2-y^2} 1 dz dA_{xy}$$

and we should use cylindrical because of the disc domain,

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r}^{6-r^2} r dz dr d\theta$$

and now we compute,

$$= \int_0^{2\pi} \int_0^2 r(6 - r^2 - r) dr d\theta = \int_0^{2\pi} \left[3r^2 - \frac{r^4}{4} - \frac{r^3}{3}\right]\Big|_0^2 d\theta$$

this integral is θ-independent,

$$= 2\pi \left[12 - 4 - \frac{8}{3}\right] = 2\pi \frac{24 - 8}{3} = \frac{32\pi}{3}.$$

Problem 4

Let the vector field $F(x, y, z) =< x, y, z >$. Let C be parameterized by $r(t) =< x(t), y(t), z(t) >$ from time $a \leq t \leq b$, where also $|r(t)| =$ constant. Show that $\int_C F \cdot dr = 0$.

Proof. First, following definitions, for the line integral, we need $r'(t) =< x'(t), y'(t), z'(t) >$. Then,

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt = \int_a^b [x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] dt.$$

Next, note that if $|r(t)| = \textit{constant}$, then $|r(t)|^2 = \textit{constant}$ too. Thus,

$$|r(t)|^2 = x(t)^2 + y(t)^2 + z(t)^2 = \textit{constant}.$$

Hence, looking at the inside of the integral, which looks like the derivative of $|r(t)|^2$, we see that

$$\frac{d}{dt}|r(t)|^2 = \frac{d}{dt}(\textit{constant}) = 0, \iff 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0.$$

In other words, since $|r(t)| = \textit{constant}$, $x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = 0$. This is exactly the integrand of the line integral, so we have that

$$\int_C F \cdot dr = \int_a^b [x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] dt = \int_a^b 0 dt = 0.$$

(And we are done with the proof)

□

Problem 5

Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$.
a) Show that $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy$

→ Mainly, $e^{-x^2-y^2} = e^{-x^2} e^{-y^2}$. Then, the integral separates,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = \int_{-\infty}^{\infty} e^{-y^2} dy \int_{-\infty}^{\infty} e^{-x^2} dx = I \cdot I = I^2.$$

b) Use polar to evaluate (a) and hence find the value of I.

$\rightarrow$ In polar, if we do x, y from $-\infty$ to ∞, this means $0 \le r \le \infty$ and $0 \le \theta \le 2\pi$. So, in polar:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \overset{\theta-indep}{=} \int_0^{\infty} 2\pi r e^{-r^2} dr$$
$$\overset{u=r^2,\ du=2rdr}{=} \int_0^{\infty} \pi e^{-u} du = -\pi e^{-u}\Big|_0^{\infty}$$
$$= \pi(e^0 - \lim_{u\to\infty} e^{-u}) = \pi(1-0) = \pi.$$

Since $I^2 = \pi$, we infer $I = \sqrt{\pi}$. Don't forget that $dV = rdzdrd\theta$ in cylindrical!

Problem 6

I want to write this problem in different notation:
Let E be a region in the xy plane. Let $T(x, y) = (x + y, x - y)$. Show that the area of $T(E) = \{T(x, y) : (x, y) \in E\}$ is twice the area of E.

Proof. First. the idea is that T induces a change of variable $u = x+y, v = x-y$ and $T(E)$ is the uv domain that E transforms into. If we were to compute the area of E in these new uv coordinates, we would need to (i) change the domain and (ii) get the Jacobian first. The domain E changes to $T(E)$ so we are done with step (i). For the Jacobian, we first need to solve x, y as functions of u, v. One way is to add both equations to get that $u + v = 2x$, and then subtract the equations to get $u - v = 2y$ so we have

$$x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v).$$

Then,

$$J = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = -1/4 - 1/4 = -1/2.$$

Therefore, if we were to calculate the area of E as an integral in uv variables,

$$Area(E) = \iint_E 1 dA = \iint_{T(E)} 1|J| dudv = \iint_{T(E)} \frac{1}{2} dudv.$$

But the right side is exactly $\frac{1}{2} \cdot Area(T(E))$. Hence, $Area(E) = \frac{1}{2} Area(T(E))$ or in other words, $Area(T(E)) = 2 \cdot Area(E)$.

$\square$

Problem 7

Evaluate $\int_C ydx + zdy + xdz$ where C is parameterized by $x = \sqrt{t}, y = t, z = t^2$ from $1 \le t \le 4$.
$\rightarrow$ First, we need the derivatives, $x'(t) = \frac{1}{2\sqrt{t}}, y'(t) = 1, z'(t) = 2t$. Thus,

$$\int_C ydx + zdy + xdz = \int_1^4 t\left(\frac{1}{2\sqrt{t}} dt\right) + t^2(1dt) + \sqrt{t}(2tdt)$$
$$= \int_1^4 \frac{\sqrt{t}}{2} dt + t^2 dt + 2t^{3/2} dt$$
$$= \frac{t^{3/2}}{3} + \frac{t^3}{3} + \frac{4t^{5/2}}{5}\Big|_1^4$$

which we can evaluate if necessary.

Problem 8

Evaluate $\iiint_E ye^{x^2+y^2+z^2}dV$ where E is the portion of the unit ball $x^2+y^2+z^2 \leq 1$ in the 1st octant.

$\to$ We have spherical objects, so we should use spherical coordinates. First, the function becomes

$$ye^{x^2+y^2+z^2} = \rho \sin\theta \sin\phi \cdot e^{\rho^2}.$$

Next, to get the right bounds for the 1st octant, where $x, y, z > 0$, we see that $0 \leq \rho \leq 1$ from the unit ball equation (which reads as $\rho^2 \leq 1$), that $0 \leq \theta \leq \pi/2$ because we have $x, y > 0$, and lastly that $0 \leq \phi \leq \pi/2$ because $z \geq 0$. Now our integral reads as (don't forget what dV is in spherical!):

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{1} \rho\sin\theta\sin\phi\ e^{\rho^2}\cdot\rho^2\sin\phi d\rho d\theta d\phi = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{1}\rho^3 e^{\rho^2}\sin\theta\sin^2\phi d\rho d\theta d\phi.$$

The e^{ρ^2} is problematic, so we need to set $w = \rho^2, dw = 2\rho d\rho$ as a substitution. Then, the idea is that $e^{\rho^2} = e^w$, and one power of the ρ^3 term cancels from $dw = 2\rho d\rho$ so we are just left with $\frac{\rho^2}{2} = w/2$. So we have

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{w=0}^{1} we^w \sin\theta\sin^2\phi\ \frac{dw}{2}d\theta d\phi$$

where now integrating by parts, with $u = w, dv = e^w dw$,

$$= \frac{1}{2}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin^2\phi\sin\theta\left[we^w\Big|_0^1 - \int_0^1 e^w dw\right]d\theta d\phi$$

$$= \frac{1}{2}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin^2\phi\sin\theta[(e^1-0)-(e^1-1)]d\theta d\phi \quad (used\ 1 = e^0)$$

$$= \frac{1}{2}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin^2\phi\sin\theta d\theta d\phi = \frac{1}{2}\int_{\phi=0}^{\pi/2}\frac{1}{2}(1-\cos 2\phi)\cdot(-\cos\theta\Big|_0^{\pi/2})d\phi$$

$$= \frac{1}{4}\int_0^{\pi/2}(1-\cos 2\phi)d\phi = \frac{1}{4}\left[\phi - \frac{\sin 2\phi}{2}\right]\Bigg|_0^{\pi/2} = \frac{\pi}{8}.$$